Today's Topics

- Basic probability
- Factorials, combinations and permutations
- Fun with homework!

Basic probability

We already talked about basic probability. Eventually I'll write up notes for it. For now, read Schacht pages 102–9, or take notes.

Factorials

While figuring out probabilities by writing out all of the possibilities works, it can be a rather unwieldy approach in all but the simplest examples.

To make these problems easier, we use *permutations* and *combinations*. Both of these shortcuts, which I'll explain in a minute or two, rely on something else: the *factorial*.

Factorials are a simple way to express large multiplications, much like $\sum$ is a simple way to express adding up lots of things. We use the exclamation point (!) to symbolize a factorial (e.g. $3!$).

Factorial example

To figure out a factorial ($n!$), we multiply all the integers from 1 to n. Formally:

$$n! = \begin{cases} \prod_{i=1}^{n} i & n \geq 1 \\ 1 & n = 0 \end{cases}$$

For example, $3! = 3 \times 2 \times 1 = 6$.

Factorial trivia

Factorials can become very big, very quickly; most calculators can't figure out factorials above $14!$ exactly, because of limited precision, and will produce an error message above $38!$ or so (they "overflow" when they can't handle a number above 10^{100} or so).

$0!$ is defined as 1.

When we figure out combinations and permutations, we can simplify factorials and make our lives easier. Often we will be asked to divide one factorial by another.

Another factorial example

For example, say we want to figure out $18!/16!$. While our calculators can provide an answer, it may no longer be precise. So, we need to figure it out another way.

If our calculators were good enough (and my computer's is):

$$\frac{18!}{16!} = \frac{18 \times 17 \times 16 \times 15 \times \cdots \times 2 \times 1}{16 \times 15 \times \cdots \times 2 \times 1} = \frac{6402373705728000}{20922789888000} = 306.$$

Unfortunately, most pocket calculators can't handle this, and they'll give you a wrong answer. So how do we simplify this?

Simplifying factorial division

If we look back at the division before, we can see that most of the terms in the numerator and denominator are the same. Thus, we can cancel out the identical terms and simplify the equation:

$$\frac{18!}{16!} = \frac{18 \times 17 \times 16 \times 15 \times \cdots \times 2 \times 1}{16 \times 15 \times \cdots \times 2 \times 1} = \frac{18 \times 17}{1} = 18 \times 17 = 306.$$

Permutations

A *permutation* is like an "and" statement in basic probability: the number of ways that two or more events occurring in a *certain order* and *without replacement* can happen.

For example, we might want to ask ourselves the probability of drawing a Heart and then a Spade from a set of four cards (one of each suit).

Permutation example (cont'd)

On the first draw, one of the four cards is a Heart. If we draw a Heart on the first draw, one of the three remaining cards is a Spade. Thus, we have:

$$\frac{1}{4} \times \frac{1}{3} = \frac{1}{12} = .083\overline{3}$$

Using the permutation formula

Instead, we can use the following *permutation formula*:

$$nPm = \frac{N!}{(N-M)!}$$

In this formula, N is the number of elements considered (i.e., the number of cards in the deck), and M is the number of elements that must appear in a certain order.

Permutation formula (cont'd)

In this case:

$$nPm = \frac{4!}{(4-2)!} = \frac{4 \times 3 \times 2 \times 1}{2 \times 1} = 4 \times 3 = 12.$$

Permutations show all the ways

The number 12 means there are twelve possible ways to draw two cards from a deck of four. Since we are only interested in one of these possibilities, this means the probability of finding the one way we want out of the 12 possibilities is $1/12$. This is the same answer we got above.

Let's think about another possibility. Say we are interested in the outcome of a 12-team basketball tournament (like the SEC championship); specifically, we want to know the odds of picking the winner and runner-up. Assuming we don't know the seedings and each team is equally likely to win, $N = 12$ and $M = 2$.

Example (cont'd)

$$nPm = \frac{12!}{(12-2)!} = \frac{12 \times 11 \times 10 \times 9 \times \cdots \times 2 \times 1}{10 \times 9 \times \ldots 2 \times 1} = 12{\times}11 = 132.$$

Since there are 132 possible permutations, the odds of us picking the one possible winner and runner-up are one in 132 or $1/132 = .0075\overline{75}$. The moral of this story: don't bet on basketball.

Combinations: order doesn't matter

In a lot of other situations, we don't care so much about the order as we do about what items are chosen. For example, in Blackjack, it usually doesn't matter which order you get a 10-card and an ace; either way, it's 21. Similarly, in poker the order of the cards is less important than what cards you have.

When we have problems like this, we use *combinations* instead of permutations. (Confusingly, in normal life a lot of things that are actually permutations are called "combinations": for example, a bicycle lock's combination is really a permutation.)

Example from Schacht

Let's think about the example that starts on page 113 of Schacht: the probablility of exactly 2 of 5 children being male. (This is the same problem as the probability of 2 of 5 coin tosses being heads.)

For each child, there are two possible outcomes (male or female). So, with five children, there are $2^5 = 32$ possible outcomes.

How many of these 32 possible outcomes are desired? That is, how many of these outcomes will have 2 male and 3 female children? If we compile a table (see the top of page 114), we'll find that there are 10 possible ways to have 2 male and 3 female children out of 5.

The combination formula

So, this means that 10 of 32 outcomes result in 2 male and 3 female children, or the probability of this outcome happening is $10/32 = .3125$.

Rather than making a table, we can figure out the number of outcomes using this formula (the *combinations* formula):

$$nCm = \frac{N!}{M!(N-M)!}$$

Note that this formula looks similar to the permutations formula, and N and M have similar meanings. However, the $M!$ in the denominator is new.

Solving this problem with the formula

So, instead of writing the possibilities out by hand, we can use the nCm formula to figure them out. Since we are interested in the probability of 2 male births occurring out of 5 births, $N = 5$ and $M = 2$. Substituting into the formula:

$$\begin{aligned} nCm = \frac{N!}{M!(N-M)!} &= \frac{5!}{2!(5-2)!} \\ &= \frac{5 \times 4 \times 3 \times 2 \times 1}{(2 \times 1)(3 \times 2 \times 1)} \\ &= \frac{5 \times 4}{2 \times 1} = \frac{20}{2} = 10 \end{aligned}$$

A harder problem

Now, let's think about a harder problem: finding the probability of having 5 male children and 2 female children out of seven consecutive births. Now, there are 2^7 or 128 possible combinations of children. How many of those would involve exactly 5 male children? With $N = 7$ and $M = 5$:

$$nCm = \frac{7!}{5!(7-5)!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1(2 \times 1)} = \frac{7 \times 6}{2 \times 1} = \frac{42}{2} = 21.$$

So, 21 of 128 possible combinations will produce 5 of 7 children being male; or the probability of this occurring is $21/128 = .1640$.

Even more complex

Now, let's think about a different problem: what are the odds of flipping three or fewer heads out of eight consecutive coin flips. (Or, what are the odds of having three or fewer male children out of eight children? – the problems are the same.) This sounds a lot like our Z-score problems from Chapter 6: in fact, it is virtually the same, except now we have *discrete* rather than *continuous* outcomes.

This problem breaks down into four parts: what are the odds of having 0 heads, 1 head, 2 heads, and three heads. We then take the four parts and put them together.

(Work through example.)

Homework

Read through the discussion of probability distributions (pages 117–19). We will discuss it briefly next time, too.

Answer questions 2 and 3 on pages 120–21.